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A RELATION BETWEEN TWO BIHARMONIC GREEN'S FUNCTIONS ON RIEMANNIAN MANIFOLDS

BY

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ABSTRACT. The biharmonic Green's function γ whose values and Laplacian are identically zero on the boundary of a region and the biharmonic Green's function Γ whose values and normal derivative vanish on the boundary originated in the investigation of thin plates whose edges are simply supported or clamped, respectively. A relation between these two biharmonic Green's functions known for planar regions is extended to Riemannian manifolds thereby establishing that any Riemannian manifold for which γ exists must also carry Γ.

Introduction. In a paper by N. Aronszajn, the integral representation of Γ given by

$$\Gamma(x,y) = \int_D g(x,\xi)g(y,\xi)\,d\xi - \int_D g(x,\xi)k(\xi,\eta)g(y,\eta)\,d\xi\,d\eta$$

is credited to S. Zaremba (see [1, p. 387]) where g is the harmonic Green's function, k is the reproducing kernel for the square integrable harmonic functions and D is a regular subregion of the plane. (For physical interpretations of γ and Γ alluded to in the abstract, see e.g. [2, Chapter IV, particularly pp. 236, 242]. An informative discussion relating k and Γ for plane regions is given in [3] and [4, pp. 265–272].) In the present paper, we note that in this representation of Γ, the first term is none other than γ, and the second term is the reproducing kernel K for the biharmonic potentials with square integrable Laplacians w.r.t. an appropriate inner product $(\, , \,)$. Also, in extending this relation between γ and Γ to Riemannian manifolds it is more natural to consider it as a representation of γ. Explicitly, we prove

THEOREM 1. *On an arbitrary Riemannian manifold, if γ exists, then K and Γ also exist. Furthermore, K and Γ are orthogonal w.r.t. $(\, , \,)$ and $\gamma = K + \Gamma$.*

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1. **Definitions.** Let R denote a Riemannian manifold, Δ its Laplace-Beltrami operator, and g_x^Ω the harmonic Green's function for a regular subregion $\Omega \subset R$ with pole $x \in \Omega$. Expressed by γ_x^Ω the biharmonic Green's function of Ω satisfying the boundary conditions

$$\gamma_x^\Omega = 0, \qquad \Delta\gamma_x^\Omega = 0 \quad \text{on } \partial\Omega,$$

and by Γ_x^Ω the biharmonic Green's function of Ω satisfying

$$\Gamma_x^\Omega = 0, \qquad \frac{\partial}{\partial\nu}\Gamma_x^\Omega = 0 \quad \text{on } \partial\Omega,$$

and where each biharmonic Green's function has a fundamental singularity at x, i.e. $\Delta\gamma_x^\Omega - g_x^\Omega$ and $\Delta\Gamma_x^\Omega - g_x^\Omega$ each can be extended to a function harmonic in all of Ω. In the above, $\partial/\partial\nu$ refers to the normal derivative and ∂ is the boundary operator.

If $\{\Omega\}$ is an exhaustion of R by regular subregions, the biharmonic Green's functions γ_x, Γ_x of R are said to exist provided the limits $\gamma_x = \lim \gamma_x^\Omega$ and $\Gamma_x = \lim \Gamma_x^\Omega$ as $\Omega \nearrow R$ exist and are finite on $R - \{x\}$. (Throughout this paper, if there is no reference to any region, it will be understood that the region shall be the entire manifold R, e.g. $\gamma_x = \gamma_x^R$, $\Gamma_x = \Gamma_x^R$.) If γ_x (similarly Γ_x) exists for all $x \in R$, we say that R possesses the biharmonic Green's function γ (respectively Γ). The family of Riemannian manifolds void of γ or Γ is denoted by O_γ or O_Γ, respectively.

COROLLARY. $O_\Gamma \subset O_\gamma$.(2)

2. **The biharmonic Green's function** γ. The class of parabolic manifolds (manifolds R void of the harmonic Green's function g, i.e. $g_x = \lim_{\Omega \nearrow R} g_x^\Omega$ is not finite for some $x \in R$) is customarily denoted by O_G. For $R \notin O_G$, we define a family F of real valued functions on R by

$$F = \left\{ f \,\middle|\, \int_R |f(\xi)| g_x(\xi)\, d\xi \text{ is well defined and finite for all } x \in R \right\},$$

and for $f \in F$ we define the function Gf on R by

$$Gf(x) = \int_R f(\xi) g_x(\xi)\, d\xi = \langle f, g_x \rangle.$$

The G-operator is an "inverse" for Δ in the following sense:

(i) If $f \in F$ and $Gf \in C^2(R)$, then $\Delta Gf = f$.

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(2) Subsequent to the writing of this paper, the author has been informed that although presently unavailable in the literature, two alternative proofs of the relation $O_\Gamma \subset O_\gamma$ are known—both using entirely different methods from those presented here. Furthermore, it is known that $\phi < O_\Gamma < O_\gamma$ (Chung-Nakai-Ralston-Sario).

(ii) If $\varphi \in C_0^\infty$, i.e. φ is C^∞ and has compact support in R, then $G\Delta\varphi = \varphi$.

(For the proof of (i) see e.g. Sario-Wang-Range [9], and for the proof of (ii) merely apply Green's identity to g and φ.)

THEOREM 2. *If γ_x exists on R, $x \in R$, then $R \notin O_G$ and*

$$\gamma_x(y) = \int_R g_x(\xi) g_y(\xi)\, d\xi \quad \textit{for all } y \in R.$$

PROOF. By the Monotone Convergence Theorem, it suffices to show that for each regular subregion Ω, $x, y \in \Omega$, $\gamma_x^\Omega(y) = \int_\Omega g_x^\Omega(\xi) g_y^\Omega(\xi)\, d\xi$. Set $f_x(y) = \int_\Omega g_x^\Omega(\xi) g_y^\Omega(\xi)\, d\xi$; then $f_x = 0$ on $\partial\Omega$ since $g_y^\Omega = 0$ for $y \in \partial\Omega$. Furthermore, $\Delta f_x = g_x^\Omega$. To see this, we observe that for every $\varphi \in C_0^\infty(\Omega)$,

$$\langle g_x^\Omega, \varphi\rangle_\Omega = \langle g_x^\Omega, G_\Omega \Delta\varphi\rangle_\Omega = \langle f_x, \Delta\varphi\rangle_\Omega = \langle \Delta f_x, \varphi\rangle_\Omega.$$

The first equality is just property (ii) satisfied by the G-operator; the second equality comes from an application of Fubini's Theorem, and the last equality utilizes Green's identity. From $\Delta f_x = g_x^\Omega$, we see that f_x has a biharmonic singularity at x, and $\Delta f_x = 0$ on $\partial\Omega$. Hence, f_x satisfies the conditions that uniquely define γ_x^Ω, i.e.

$$\gamma_x^\Omega(y) = f_x(y) = \int_\Omega g_x^\Omega(\xi) g_y^\Omega(\xi)\, d\xi.$$

COROLLARY 1. *γ is positive and symmetric.*

COROLLARY 2. *If γ_x exists for some $x \in R$, then γ_x exists for all $x \in R$.*

PROOF. For an arbitrary $y \in R$, we must show that $\gamma_y < \infty$ assuming $\gamma_x < \infty$ for some $x \in R$. As just seen, the existence of γ_x for some x implies the existence of g_x for all x. Let Ω be a regular subregion containing both x and y. For $\xi \in R$ and distinct from x and y, let $C_1(\xi) = \langle g_y, g_\xi\rangle_\Omega / \langle g_x, g_\xi\rangle_\Omega$, $m = \min g_x$ and $M = \max g_y$ on $\partial\Omega$. We then have

$$\begin{aligned}\gamma_y(\xi) &= \langle g_y, g_\xi\rangle_\Omega + \langle g_y, g_\xi\rangle_{R-\Omega} \\ &\leqslant C_1(\xi)\langle g_x, g_\xi\rangle_\Omega + (M/m)\langle g_x, g_\xi\rangle_{R-\Omega} \leqslant C(\xi)\gamma_x(\xi)\end{aligned}$$

where $C(\xi) = \max\{C_1(\xi), M/m\} < \infty$.

3. **Square integrable harmonic functions.** Let $HL^2(R)$ denote the square integrable harmonic functions on a Riemannian manifold R, and let $\|h\| = \langle h, h\rangle^{1/2}$ for $h \in HL^2(R)$.

THEOREM 3. *For an arbitrary Riemannian manifold R, $HL^2(R)$ is a Hilbert space. Furthermore, there exists a positive function M on R satisfying*

(1) $$|h| \leqslant M\|h\| \quad \textit{for all } h \in HL^2(R)$$

and for which $M_E = \sup_{x\in E} M(x) < \infty$ *for every compact* $E \subset R$.(3)

PROOF. We first consider the existence of M together with the finiteness of M_E. Given compact E, let Ω be a regular subregion containing E. For $x \in E$, $c > 0$, let $A_c(x)$ be the annular region

$$A_c(x) = \{\xi \in \Omega | g_x^\Omega(\xi) \leqslant c\} \quad \text{and} \quad M_1 = \sup_{x\in E, \xi \in A_c(x)} |\operatorname{grad}_\xi g_x^\Omega(\xi)|.$$

The finiteness of M_1 is a consequence of the continuity of $g_x^\Omega(\xi)$ and $|\operatorname{grad}_\xi g_x^\Omega(\xi)|$ on $\Omega \times \Omega$-diagonal and the fact that $\sup_{x\in E, \xi\in A_c(x)} g_x^\Omega(\xi) = c$. We think of $A_c(x)$ as being composed of a collection of level surfaces $\{S_d(x)\}_{0<d\leqslant c}$ where $S_d(x) = \{\xi \in \Omega | g_x^\Omega(\xi) = d\}$. If α is a flow line joining S_{d_1} to S_{d_2}, $0 \leqslant d_1 < d_2 \leqslant c$, we have

$$d_2 - d_1 = \int_\alpha |\operatorname{grad}_\xi g_x^\Omega(\xi)| \, dL_\xi \leqslant M_1 (\text{length}\, \alpha)$$

where dL_ξ refers to arc length. Hence, $(d_2 - d_1)/M_1 \leqslant \text{length}\, \alpha$. From this along with

$$|h(x)| \leqslant \int_{S_d} \left| \frac{\partial}{\partial \nu_\xi} g_x^\Omega(\xi) \right| \cdot |h(\xi)| \, dS_\xi, \qquad x \in E, 0 \leqslant d \leqslant c,$$

it follows that

$$|h(x)| \frac{c}{M_1} \leqslant M_1 \int_{A_c(x)} |h(\xi)| \, dV_\xi.$$

Here, dS_ξ is the surface element and dV_ξ is the volume element. Thus, by Schwarz we obtain

$$|h(x)| \leqslant (M_1^2/c)\sqrt{\operatorname{vol}\Omega}\, \|h\| \quad \text{for all } x \in E, h \in HL^2.$$

The existence of M and the finiteness of M_E is now clear.

To see that HL^2 is a Hilbert space, let $\{h_n\}$ be Cauchy in HL^2. By the first part of this proof just completed,

$$|h_n(x) - h_m(x)| \leqslant M_E \|h_n - h_m\|, \qquad x \in E.$$

Hence there exists h harmonic on R for which $h_n \to h$ uniformly on compact subsets of R. In particular, $\|h - h_n\|_E \to 0$ as $n \to \infty$. Also, $\{\|h_n\|\}$ is bounded

(3) The reader might find it enlightening to compare the first part of this proof with an inequality given in [3, p. 503].

since $\{h_n\}$ is Cauchy. We conclude that $h \in HL^2$ from the inequality

$$\|h\|_E \leqslant \|h - h_n\|_E + \|h_n\|$$

by taking the limit as $n \to \infty$ and then the supremum over all compact $E \subset R$. To see that $h_n \to h$ in norm, we consider the inequality

$$\|h - h_n\| \leqslant \|h - h_n\|_E + \|h_N - h_n\| + \|h_N\|_{R-E} + \|h\|_{R-E}.$$

Regarding the R.H.S., choose N sufficiently large so that the second term is $\leqslant \varepsilon/4$ for all $n \geqslant N$, then choose E so large that the sum of the last two terms $\leqslant \varepsilon/2$, and finally take $n \geqslant N$ and large enough that the first term $\leqslant \varepsilon/4$.

We restate Theorem 3 in an equivalent form.

THEOREM 3′. *For an arbitrary Riemannian manifold R, $HL^2(R)$ is a Hilbert space, and there exists a unique symmetric reproducing kernel $k \in HL^2(R)$ satisfying $h = \langle h,k\rangle$ for all $h \in HL^2(R)$. Also, $k_E = \sup_{x\in E} k_x(x) < \infty$ for each compact $E \subset R$.*

That Theorem 3′ implies Theorem 3 is clear. Conversely, the existence of k_x is assured by the Riesz representation theorem for bounded functionals defined on a Hilbert space, and by (1) which says, for every $x \in R$, evaluation is a bounded functional on HL^2. That k_E is finite is seen by substituting k_x into (1) thereby obtaining $k_E \leqslant M_E^2$. The symmetry and uniqueness of k is confirmed in the usual manner.

LEMMA 1. *$k_x(y)$ is continuous on $R \times R$.*

PROOF. Fix $x_0, y_0 \in R$ and consider the inequality

$$|k_y(x) - k_{y_0}(x_0)| \leqslant |k_y(x) - k_{y_0}(x)| + |k_{y_0}(x) - k_{y_0}(x_0)|, \qquad x, y \in R.$$

On the R.H.S., the second term offers no difficulty since k_{y_0} is continuous, in fact harmonic, and we direct our attention to the first term.

Let U, V be regular subregions of R. By Schwarz $|k_x(y)|^2 \leqslant k_x(x)k_y(y)$ so that by Theorem 3′ k is bounded on $U \times V$. Consequently, there is no harm in assuming that k is positive on $U \times V$. Let U_1 be a regular subregion whose closure $\overline{U}_1$ is contained in U. For $x \in \overline{U}_1, y, y_0 \in V$, we have

$$k_y(x) - k_{y_0}(x) = \int_{\partial U} (k_y(\xi) - k_{y_0}(\xi)) \frac{\partial}{\partial \nu_\xi} g_x^U(\xi)\, dS_\xi.$$

By the continuity of $\partial g_x^U(\xi)/\partial \nu_\xi$ on $\overline{U}_1 \times \partial U$,

$$(2) \qquad |k_y(x) - k_{y_0}(x)| \leqslant \text{const} \int_{\partial U} |k_y(\xi) - k_{y_0}(\xi)|\, dS_\xi, \qquad x \in \overline{U}_1.$$

By Harnack's inequality there exists $c > 0$ such that

$$0 < k_y(\xi) = k_\xi(y) < ck_\xi(y_0) = ck_{y_0}(\xi)$$

for all $y \in \overline{V}_1 \subset V$, $\xi \in \partial U$. Therefore,

$$|k_y(\xi) - k_{y_0}(\xi)| \leqslant (c+1)k_{y_0}(\xi),$$

where the R.H.S. is integrable over the ∂U. Hence the Lebesgue Dominated Convergence Theorem applies to (2), and the proof is herewith complete.

Let Ω denote a regular subregion of R and Ω' another regular subregion or possibly $\Omega' = R$.

LEMMA 2. *For every* $x \in \Omega \subset \Omega'$,

$$0 \leqslant \|k_x^\Omega - k_x^{\Omega'}\|_\Omega^2 \leqslant k_x^\Omega(x) - k_x^{\Omega'}(x).$$

PROOF. Expand $\langle k_x^\Omega - k_x^{\Omega'}, k_x^\Omega - k_x^{\Omega'}\rangle_\Omega$ and employ the reproducing properties of k_x^Ω and $k_x^{\Omega'}$.

REMARK. Taking Ω' to be R, we obtain as an immediate consequence of Lemma 2 that

$$\|k_x^\Omega\|_\Omega^2 = k_x^\Omega(x) \searrow k_x(x) = \|k_x\|^2.$$

Thus, $k_x^\Omega \to k_x$ in L^2 norm which together with (1) of Theorem 3 says that the convergence is also uniform on compacta.

4. A reproducing kernel for biharmonic potentials with square integrable Laplacians. If $R \notin O_\gamma$, then $HL^2 \subset F$. To see this, recall that in the proof of Corollary 2 in §2, for fixed $x, y \in \Omega$, $x \neq y$, $g_y \leqslant (M/m)g_x$ on $R - \Omega$ so that

$$\|g_y\|_{R-\Omega} \leqslant (M/m)\langle g_y, g_x\rangle_{R-\Omega} < \gamma_x(y) < \infty. \tag{3}$$

Hence, for $h \in HL^2(R)$,

$$\begin{aligned}\int_R |h(\xi)| g_y(\xi)\, d\xi &= \langle |h|, g_y\rangle_\Omega + \langle |h|, g_y\rangle_{R-\Omega}\\ &\leqslant \langle |h|, g_y\rangle_\Omega + \|h\|\,\|g_y\|_{R-\Omega} < \infty.\end{aligned}$$

If $R \notin O_\gamma$, by the biharmonic potentials with square integrable Laplacians, we mean $GHL^2 = \{Gh | h \in HL^2\}$. We define an inner product $(\ ,\ )$ on GHL^2 by

$$(u, v) = \langle \Delta u, \Delta v\rangle, \qquad u, v \in GHL^2$$

and denote the induced norm by $|||\ \ |||$.

THEOREM 4. *If $R \notin O_\gamma$, then $GHL^2(R)$ is a Hilbert space, and there exists a positive function $\mathbf{M}^R$ such that $|u| \leqslant \mathbf{M}^R |||u|||$ for all $u \in GHL^2(R)$.*

PROOF. That GHL^2 is a Hilbert space is easily seen from the fact that HL^2 is a Hilbert space.

For $x \in R$ and $c > 0$, let $U = \{\xi \in R | g_x(\xi) > c\}$. For $u \in GHL^2$, apply Green's identity to γ_x^U and $h = \Delta u$ thereby obtaining

$$G_U h(x) = \int_{\partial U} h(\xi) \frac{\partial}{\partial \nu_\xi} \gamma_x^U(\xi)\, dS_\xi .$$

From this representation together with the reasoning as given in the first part of Theorem 3, it follows that there exists $m(x) > 0$ such that $|G_U h(x)| \leqslant m^U(x) \|h\|_U$ for all $u \in GHL^2$. Note that $G_U h(x) = \langle h, g_x - c\rangle_U$ since $g_x^U = g_x - c$ on U. Consequently, $|\langle h, g_x - c\rangle_U| \leqslant m^U(x)\|h\|_U$. Therefore, we have

$$\begin{aligned} |u(x)| &\leqslant |\langle h, g_x - c\rangle_U| + |\langle h, c\rangle_U| + |\langle h, g_x\rangle_{R-U}| \\ &\leqslant m^U(x)\|h\|_U + c\sqrt{\operatorname{vol} U}\,\|h\|_U + \|g_x\|_{R-U}\|h\|_{R-U} \\ &\leqslant \mathbf{M}^R(x)|||u||| \end{aligned}$$

where

$$\mathbf{M}^R(x) = \max\{m^U(x) + c\sqrt{\operatorname{Vol} U}, \|g_x\|_{R-U}\} \tag{4}$$

is finite and independent of u.

THEOREM 4′. *If $R \notin O_\gamma$, then $GHL^2(R)$ is a Hilbert space and there exists a unique symmetric reproducing kernel $K \in GHL^2(R)$ such that $u = (u, K)$ for all $u \in GHL^2(R)$.*

THEOREM 5. *If $R \notin O_\gamma$, then*

$$K_x(y) = \int_{R \times R} g_x(\xi) k_\xi(\eta) g_y(\eta)\, d\xi\, d\eta, \qquad x, y \in R.$$

PROOF. Define h_x on R by $h_x(\xi) = Gk_\xi(x)$; then we claim that $h_x = \Delta K_x$. To establish our claim, it suffices to show that $\langle \varphi, h_x\rangle = \langle \varphi, \Delta K_x\rangle$ for all $\varphi \in C_0^\infty$. Since HL^2 is a closed subspace of L^2, there exist unique $\varphi_1 \in HL^2$, $\varphi_2 \in (HL^2)^\perp$ such that $\varphi = \varphi_1 + \varphi_2$. Here $(HL^2)^\perp$ denotes the orthogonal complement of HL^2 in L^2. Therefore,

$$\langle \varphi, h_x \rangle = \int_R \varphi(\xi) G k_\xi(x)\, d\xi = \int_R \varphi(\xi)\left(\int_R k_\xi(\eta) g_x(\eta)\, d\eta\right) d\xi$$
$$= \int_R \left(\int_R \varphi(\xi) k_\eta(\xi)\, d\xi\right) g_x(\eta)\, d\eta = \int_R \varphi_1(\eta) g_x(\eta)\, d\eta = G\varphi_1(x).$$

The first equality is just the definition of h_x, the second and last equalities come from the definition of the G-operator, the third equality is Fubini, and the fourth equality uses the reproducing property of k_η and the fact that φ_2 and k_η are orthogonal. On the other hand, by the orthogonality of φ_2 and ΔK_x, by property (i) of §2, by the definition of $(\ ,\ )$, and by the reproducing property of K_x, we have

$$\langle \varphi, \Delta K_x \rangle = \langle \varphi_1, \Delta K_x \rangle = \langle \Delta G\varphi_1, \Delta K_x \rangle = (G\varphi_1, K_x) = G\varphi_1(x),$$

which completes the proof of our claim.

Since $K_x \in GHL^2$ there exists $h \in HL^2$ such that $K_x = Gh$. However, $h_x = \Delta K_x = h$ so that $K_x = Gh_x$ which when written out is the R.H.S. of our theorem.

5. **Convergence of reproducing kernels for potentials.** In this section, we prove the following theorem.

THEOREM 6. *For $R \notin O_\gamma$, $K^\Omega \to K$ pointwise and in norm $|||\ |||$ as $\Omega \nearrow R$.*

LEMMA 3. *If $R \notin O_\gamma$, then for every $x \in R$ $\|g_x - g_x^\Omega\| \searrow 0$ as $\Omega \nearrow R$.*

PROOF. By (3) at the beginning of §4, $R \notin O_\gamma$ guarantees that $\|g_x\|_{R-\Omega} < \varepsilon/3$ for given $\varepsilon > 0$ and sufficiently large Ω. Having chosen such an Ω, choose $c > 0$ so small that $\Omega \subset \Omega'$ where $\Omega' = \{\xi \in R | g_x(\xi) > c\}$ and $c\sqrt{\operatorname{vol}\Omega} < \varepsilon/3$. Consider the inequality

$$\|g_x - g_x^{\Omega'}\| \leqslant \|g_x\|_{R-\Omega} + \|g_x^{\Omega'}\|_{R-\Omega} + \|g_x - g_x^{\Omega'}\|_\Omega.$$

Since $\|g_x^{\Omega'}\|_{R-\Omega} < \|g_x\|_{R-\Omega} < \varepsilon/3$, the sum of the first two terms on the R.H.S. is $< 2\varepsilon/3$. Furthermore, $g_x - g_x^{\Omega'}$ is harmonic on Ω' and $= c$ on $\partial\Omega'$ so that $g_x - g_x^{\Omega'} = c$ throughout Ω'. Therefore, the last term $= c\sqrt{\operatorname{vol}\Omega} < \varepsilon/3$ which completes the proof.

Regarding functions which up to now were considered to be defined only on some subregion Ω of a Riemannian manifold R, we shall find it convenient to henceforth consider them to be defined on all of R by making them $= 0$ on the complement of Ω. In particular, by setting $g^\Omega = 0$ outside of Ω, we have also extended G_Ω to be an operator on $F(R)$—explicitly, $G_\Omega f(x) = 0$, $x \in R - \Omega$, $f \in F(R)$. Not only will our notation fail to distinguish between a function defined on Ω and its trivial extension, it will continue to ignore the distinction between a function and its restriction.

Recall from the proof of Theorem 5, the function h_x given by $h_x(\xi) = Gk_\xi(x)$, and similarly define h_x^Ω by $h_x^\Omega(\xi) = G_\Omega k_\xi^\Omega(x)$. Also define $h_{\Omega,x}$ by $h_{\Omega,x}(\xi) = G_\Omega k_\xi(x)$. Considerations at the beginning of §4 assure that these functions are well defined. That $h_x \in L^2$ is clear since $h_x = \Delta K_x \in L^2$ and similarly for h_x^Ω. To show $h_{\Omega,x}$ is square integrable, we need only show that $h_x - h_{\Omega,x} \in L^2$. We note that $f = g_x - g_x^\Omega \in L^2$ by Lemma 3 and that $h_x(\xi) - h_{\Omega,x}(\xi) = \langle f, k_\xi \rangle$. Since $f = f_1 + f_2$ with $f_1 \in HL^2$ and $f_2 \in (HL^2)^\perp$, we see that $\langle f, k_\xi \rangle = f_1(\xi)$. Therefore we conclude,

$$\|h_x - h_{\Omega,x}\| = \|f_1\| \leqslant \|f_1\| + \|f_2\| = \|f\| = \|g_x - g_x^\Omega\| < \infty.$$

We have proven:

LEMMA 4. *Given $R \notin O_\gamma$, then $h_x, h_{\Omega,x} \in HL^2(R)$, and $\|h_x - h_{\Omega,x}\| \to 0$ as $\Omega \nearrow R$. In fact, $\|h_x - h_{\Omega,x}\| \leqslant \|g_x - g_x^\Omega\|$.*

LEMMA 5. *For $R \notin O_\gamma$, $\|h_x - h_x^\Omega\| \to 0$ as $\Omega \nearrow R$.*

PROOF. Since

$$\|h_x - h_x^\Omega\| \leqslant \|h_x - h_{\Omega,x}\| + \|h_{\Omega,x} - h_x^\Omega\|,$$

by Lemma 4 we need only show that $\|h_{\Omega,x} - h_x^\Omega\|_E \to 0$ as $\Omega \nearrow R$ for every compact E. By the definitions of $h_{\Omega,x}$ and h_x^Ω, the linearity of the G_Ω-operator, Theorem 4, and Lemma 2, we have for all $\Omega \supset E$,

$$\begin{aligned}\int_E (h_{\Omega,x}(\xi) - h_x^\Omega(\xi))^2 d\xi &= \int_E (G_\Omega k_\xi(x) - G_\Omega k_\xi^\Omega(x))^2 d\xi \\ &= \int_E [G_\Omega(k_\xi - k_\xi^\Omega)]^2(x)\, d\xi \leqslant (\mathbf{M}^\Omega(x))^2 \int_E \|k_\xi - k_\xi^\Omega\|_\Omega^2 d\xi \\ &\leqslant (\mathbf{M}^\Omega(x))^2 \int_E (k_\xi^\Omega(\xi) - k_\xi(\xi))\, d\xi.\end{aligned}$$

By (4) in §4, we see that $\mathbf{M}^\Omega(x) < \mathbf{M}^R(x) < \infty$. Also, by Lemmas 1 and 2, $k_\xi^\Omega(\xi)$ and $k_\xi(\xi)$ are measurable, in fact continuous, and $k_\xi^\Omega(\xi) \searrow k_\xi(\xi)$ on E so that the Monotone Convergence Theorem assures the last expression $\to 0$ as $\Omega \nearrow R$.

COMPLETION OF THE PROOF OF THEOREM 6. Subtracting and adding $\langle K_x^\Omega, K_\xi \rangle$ and by Schwarz, we obtain

$$|K_x(\xi) - K_x^\Omega(\xi)| \leqslant |||K_\xi||| \cdot |||K_x - K_x^\Omega||| + |||K_x^\Omega||| \cdot |||K_\xi - K_\xi^\Omega|||,$$

so that we need only show $K_x^\Omega \to K_x$ in norm. However, $|||K_x - K_x^\Omega||| = \|h_x - h_x^\Omega\| \to 0$ as $\Omega \nearrow R$.

6. Completion of the proof of Theorem 1. We first show that $\Gamma_x^\Omega = \gamma_x^\Omega - K_x^\Omega$ for each regular subregion Ω, $x \in \Omega$. Since K_x^Ω is biharmonic on Ω and γ_x^Ω has a biharmonic singularity at $x \in \Omega$, surely $\gamma_x^\Omega - K_x^\Omega$ is biharmonic on $\Omega - \{x\}$ and possesses a biharmonic singularity at x. It is also clear that $\gamma_x^\Omega - K_x^\Omega = 0$ on $\partial\Omega$ since each term $= 0$ on $\partial\Omega$. Using this together with Green's identity, we have

$$(5) \quad \int_{\partial\Omega} h(\xi)\frac{\partial}{\partial\nu_\xi}(\gamma_x^\Omega(\xi) - K_x^\Omega(\xi))\,dS_\xi = -\int_\Omega h(\xi)\Delta(\gamma_x^\Omega(\xi) - K_x^\Omega(\xi))\,dV_\xi$$

where h is harmonic. Since $\int_\Omega h(\xi)\Delta\gamma_x^\Omega(\xi)\,dV_\xi = G_\Omega h(x)$ and $\int_\Omega h(\xi)\Delta K_x^\Omega(\xi) = G_\Omega h(x)$, we conclude that the R.H.S. of (5) $= 0$. Applying Green's identity to the function $\equiv 1$ and $\gamma_x^\Omega - K_x^\Omega$, we see that $\int_{\partial\Omega} (\partial/\partial\nu_\xi)(\gamma_x^\Omega(\xi) - K_x^\Omega(\xi))\,dS_\xi = 0$. Hence there exists a harmonic solution to the boundary value problem $h = (\partial/\partial\nu)(\gamma_x^\Omega - K_x^\Omega)$ on $\partial\Omega$. Substituting this solution into (5), we see that $(\partial/\partial\nu)(\gamma_x^\Omega - K_x^\Omega) = 0$ on $\partial\Omega$, thereby verifying that $\Gamma^\Omega = \gamma^\Omega - K^\Omega$. Hence, if γ exists, by Theorem 6 K exists, and $\Gamma = \lim_{\Omega \nearrow R}(\gamma^\Omega - K^\Omega) = \gamma - K$.

Lastly, K and Γ are orthogonal since

$$(K_x, \Gamma_x) = \int_R \Delta K_x(\xi)(\Delta\gamma_x(\xi) - \Delta K_x(\xi))\,d\xi = K_x(x) - K_x(x) = 0.$$

In closing, I would like to hint at other applications of the methods presented. From Theorem 2, it is immediate that the existence of a positive quasiharmonic function implies the existence of γ [5]. On the other hand, it is clear that the existence of γ assures that the biharmonic functions with square integrable Laplacians possess Riesz representations [6], [9]. Since Theorem 3′ guarantees that k always exists, one can define a span whose vanishing is equivalent to the nonexistence of nonzero square integrable harmonic functions [8]. Also, K may be found useful in formulating and solving a biharmonic interpolation problem similar to one known for harmonic functions [4, pp. 275–280], [7].

Bibliography

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404. MR **14**, 479.
2. S. Bergman and M. Schiffer, *Kernel functions and elliptic differential equations in mathematical physics*, Academic Press, New York, 1953. MR **14**, 876.
3. P. Garabedian, *A partial differential equation arising in conformal mapping*, Pacific J. Math. **1** (1951), 485–524. MR **13**, 735.
4. ———, *Partial differential equations*, 2nd ed., Wiley, New York, 1967.
5. M. Nakai and L. Sario, *Quasiharmonic classification of Riemannian manifolds*, Proc. Amer. Math. Soc. **31** (1972), 165–169. MR **44** #4692.
6. ———, *Dirichlet finite biharmonic functions with Dirichlet finite Laplacians*, Math. Z. **122** (1971), 203–216. MR **45** #2616.

7. L. Sario, *Extremal problems and harmonic interpolation on open Riemann surfaces*, Trans. Amer. Math. Soc. **79** (1955), 362–377. MR **19**, 846.

8. L. Sario, M. Schiffer and M. Glasner, *The span and principal functions in Riemannian spaces*, J. Analyse Math. **15** (1965), 115–134. MR **32** #1655.

9. L. Sario, C. Wang and M. Range, *Biharmonic projection and decomposition*, Ann. Acad. Sci. Fenn. A.I. **494** (1971), pp. 1–14.

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